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NATURAL FREQUENCY CONSTRAINED OPTIMAL STRUCTURAL MODIFICATION USING A SENSITIVITY DERIVATIVE OF DYNAMICALLY CONSTRAINED MASS AND STIFFNESS MATRICES

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1. INTRODUCTION

Structural dynamic modification (SDM) problems involve local modifications with lumped masses, stiffeners, dampers and beam elements, etc., to improve the dynamic behaviour of existing structures. Many proposals have been presented earlier [1–3] which identify the optimal parameter of the system. The need for quantification of identified optimal parameters for a desired natural frequency has been stressed by Snoeys *et al*. [4]. Reanalysis techniques for finding the geometrical dimensions of structural members have been investigated by Wang [5]. The modification techniques reviewed above offer no guarantee that the chosen modification is the most appropriate. Depending upon the dynamicist's intuition and other factors, the modification may be acceptable, but not necessarily optimal. This is the reason why sensitivity analysis together with modification techniques form useful tools for the optimum modification of the dynamic behaviour of mechanical structures. These techniques, based on the calculation of derivatives of the modal parameters, provide the most effective parameter changes such that the desired dynamic behaviour can be obtained in one modification cycle. The calculation of derivatives of modal parameters in turn requires the sensitivity derivatives of the spatial parameters, namely mass and stiffness matrices of the structure [6, 7]. Usually, these matrices are dynamically condensed corresponding to the co-ordinates of interest [8].

The finite difference calculation of sensitivity derivatives is easy to implement but the accuracy of this method is greatly influenced by the choice of step size. The present work addresses itself to analytical determination of sensitivity derivatives of dynamically condensed mass and stiffness matrices of structures which are modelled with beam elements [8]. These derivatives are used to find derivatives of eigenvalues and thereby predict optimal values of modifications such that the modified structure has the desired shifted natural frequency. For a dynamicist, it is desirable that none of the natural frequencies coincide with excitation frequencies. The present note deals with shifting of an undesirable natural frequency in an optimum way. The proposed algorithm has an advantage that a shift in natural frequency and its prediction is accurate even when the magnitude of the desired frequency shift is large. A first step in this direction is the sensitivity analysis to identify effective parameters for modification. The present work utilizes the sensitivity derivatives of the natural frequency and also quantifies the effectiveness of various parameters. Further, optimal modification is approached as one of the two possible requirements.

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When a natural frequency is desired to be shifted by small values, fast single parameter optimization (FASPO) is suggested. This method is more relevant when the desirable hardware modifications need to be small in order not to substantially change the original design. Secondly, a high accuracy single parameter optimization (HASPO) method is suggested when larger shifts of unmodified natural frequency are required and a substantial change in the design parameter is allowed.

2. THEORY

The equations of motion of an undamped multi-degree-of-freedom (MDoF) system/structure can be written as

$$[\mathbf{M}]\{\ddot{\mathbf{X}}\} + [\mathbf{K}]\{\mathbf{X}\} = 0, \tag{1}$$

where $[\mathbf{K}]$ is the stiffness matrix, $[\mathbf{M}]$ is the mass matrix, $\{\ddot{\mathbf{X}}\}$ is the acceleration vector and $\{\mathbf{X}\}$ is the displacement vector. In general, the global matrices $[\mathbf{K}]$ and $[\mathbf{M}]$ are dynamically condensed/reduced to the desired degrees of freedom and called system matrices [9–14].

2.1. *Dynamically condensed mass and stiffness matrices*

The global stiffness and mass matrices for a structure having *n* beam elements can be written as

$$[\mathbf{K}_G] = [\mathbf{R}]^T[\mathbf{K}_L][\mathbf{R}], \qquad [\mathbf{M}_G] = [\mathbf{R}]^T[\mathbf{M}_L][\mathbf{R}], \tag{2, 3}$$

where $[\mathbf{M}_L]$, $[\mathbf{K}_L]$ are global mass and stiffness matrices respectively and $[\mathbf{R}]$ is the connection matrix which can be constructed as explained in several texts on FEM [9–14].

The global matrices are invariably dynamically condensed for the co-ordinates of interest (master co-ordinates, i.e., retained DoF). The dynamic condensation technique due to Guyan [15] is used to obtain dynamically condensed matrices as follows.

Matrices $[\mathbf{K}_G]$ and $[\mathbf{M}_G]$ can be written in partitioned form as

$$[\mathbf{K}_G] = \left[\begin{array}{c|c} \mathbf{K}_G AA & \mathbf{K}_G AB \\ \hline \mathbf{K}_G BA & \mathbf{K}_G BB \end{array}\right], \qquad [\mathbf{M}_G] = \left[\begin{array}{c|c} \mathbf{M}_G AA & \mathbf{M}_G AB \\ \hline \mathbf{M}_G BA & \mathbf{M}_G BB \end{array}\right]. \tag{4, 5}$$

The reduced co-ordinates system matrices viz stiffness and mass matrices can be written as [14]

$$[\mathbf{K}_G]^R = [\mathbf{K}_G AA] - [\mathbf{K}_G AB] \cdot [\mathbf{K}_G BB]^{-1} \cdot [\mathbf{K}_G BA], \tag{6}$$

$$\begin{aligned}[\mathbf{M}_G]^R = {} & [\mathbf{M}_G AA] - [\mathbf{M}_G AB] \cdot [\mathbf{K}_G BB]^{-1} \cdot [\mathbf{K}_G BA] - [\mathbf{K}_G AB] \cdot [\mathbf{K}_G BB]^{-1} \cdot [\mathbf{M}_G BA] \\ & + [\mathbf{K}_G AB] \cdot [\mathbf{K}_G BB]^{-1} \cdot [\mathbf{M}_G BB] \cdot [\mathbf{K}_G BB]^{-1} \cdot [\mathbf{K}_G BA]. \end{aligned} \tag{7}$$

In those structures where no kinetic energy is associated with co-ordinates other than those of interest, $[\mathbf{M}_G BA]$ and $[\mathbf{M}_G BB]$ are zero. Small errors in the computation of eigenvalues may occur when these co-ordinates have a small contribution of kinetic energy [14]. In such cases equation (7) can be written as

$$[\mathbf{M}_G]^R = [\mathbf{M}_G AA]. \tag{8}$$

Equations (6) and (8) are the condensed stiffness and mass matrices of the structure.

2.2. *Sensitivity derivatives of condensed system matrices*

Differentiating equations (6) and (8) with respect to a design parameter p, the sensitivity derivatives of system matrices can be written as

$$\partial[\mathbf{K}_G]^R/\partial p = \partial[\mathbf{K}_G AA]/\partial p - \partial\{[\mathbf{K}_G AB]\cdot[\mathbf{K}_G BB]^{-1}\cdot[\mathbf{K}_G BA]\}/\partial p, \tag{9}$$

$$\partial[\mathbf{M}_G]^R/\partial p = \partial[\mathbf{M}_G AA]/\partial p. \tag{10}$$

Therefore equation (9) can be written in expanded form as

$$\begin{aligned}\partial[\mathbf{K}_G]^R/\partial p = {} & \partial[\mathbf{K}_G AA]/\partial p - \partial[\mathbf{K}_G AB]/\partial p\cdot[\mathbf{K}_G BB]^{-1}\cdot[\mathbf{K}_G BA]\\ & - [\mathbf{K}_G AB]\cdot[\mathbf{K}_G BB]^{-1}\cdot\partial[\mathbf{K}_G BA]/\partial p\\ & - [\mathbf{K}_G AB]\cdot\partial[\mathbf{K}_G BB]^{-1}/\partial p\cdot[\mathbf{K}_G BA].\end{aligned} \tag{11}$$

From matrix operations it is known that

$$\partial[\mathbf{K}_G BB]^{-1}/\partial p = -[\mathbf{K}_G BB]^{-1}\cdot\partial[\mathbf{K}_G BB]/\partial p\cdot[\mathbf{K}_G BB]^{-1}.$$

Therefore equation (11) can be written as

$$\begin{aligned}\partial[\mathbf{K}_G]^R/\partial p = {} & \partial[\mathbf{K}_G AA]/\partial p - [\mathbf{X}]\cdot\partial[\mathbf{K}_G BA]/\partial p - \partial[\mathbf{K}_G AB]/\partial p\cdot[\mathbf{Y}]\\ & + [\mathbf{X}]\cdot\partial[\mathbf{K}_G BB]/\partial p\cdot[\mathbf{Y}],\end{aligned} \tag{12}$$

where

$$[\mathbf{X}] = [\mathbf{K}_G AB]\cdot[\mathbf{K}_G BB]^{-1}, \qquad [\mathbf{Y}] = [\mathbf{K}_G BB]^{-1}\cdot[\mathbf{K}_G BA]. \tag{13}$$

The analytical derivatives of the reduced co-ordinates mass and stiffness matrices can be derived using equations (10) and (12). To compute these derivatives, the computation of sub-matrices $[\mathbf{X}]$ and $[\mathbf{Y}]$ along with derivatives of sub-matrices $[\mathbf{K}_G AA]$, $[\mathbf{K}_G AB]$, $[\mathbf{K}_G BA]$, $[\mathbf{K}_G BB]$ and $[\mathbf{M}_G AA]$ are required. The matrices $[\mathbf{X}]$ and $[\mathbf{Y}]$ are obtained from equation (13). While computing these matrices, i.e., $[\mathbf{X}]$ and $[\mathbf{Y}]$ it can be observed that the constituent matrices $[\mathbf{K}_G BB]$ and $[\mathbf{K}_G BA]$ are available in the database at the modelling stage of the structure and no extra computation is required. The derivatives of sub-matrices $[\mathbf{K}_G AA]$, $[\mathbf{K}_G AB]$, $[\mathbf{K}_G BB]$, $[\mathbf{K}_G BA]$ and $[\mathbf{M}_G AA]$ with respect to the design parameter p can be obtained as follows. If the parameter p belongs to the rth beam element, the derivative of the global stiffness matrix in equation (2) with respect to p can be written assuming the beam elements are parallel to one of the global co-ordinate of the system. Further, it is also assumed that parameter p is not the length of the beam element. Therefore, the derivative of the global stiffness matrix can be written as

$$\partial[\mathbf{K}_G]/\partial p = [\mathbf{R}]^T \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & \partial[\mathbf{k}_{er}]/\partial p & \\ & & & 0 \end{bmatrix} [\mathbf{R}]. \tag{14}$$

Similarly, the derivative of the global mass matrix from (3) can be obtained as

$$\partial[\mathbf{M}_G]/\partial p = [\mathbf{R}]^T \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & \partial[\mathbf{m}_{er}]/\partial p & \\ & & & 0 \end{bmatrix} [\mathbf{R}], \tag{15}$$

where $\partial[\mathbf{k}_{er}]/\partial p$ and $\partial[\mathbf{m}_{er}]/\partial p$ are the derivatives of the rth beam element's stiffness and mass matrix respectively with respect to the parameter p (depth, width, etc.). A structural beam element is shown in Figure 1. The forces and displacements are shown for the positive stiffness co-ordinate system. The stiffness matrix and the consistent mass matrix for the beam element neglecting shear deformations can be obtained from several texts [9–12]. The derivative of these element stiffness and mass matrices with respect to geometrical parameters, namely depth, width for a rectangular cross-section beam element, can be obtained analytically [8]. These elemental system matrices' derivatives in turn are substituted in equation (14) and (15) to find the derivatives of global stiffness and mass matrices.

Further, derivatives of these global matrices can also be obtained using equations (4, 5). By equating the above two equivalent derivatives of system matrices, the derivatives of submatrices $\mathbf{K}_G AA$, $\mathbf{K}_G AB$, $\mathbf{K}_G BA$, $\mathbf{K}_G BB$ and $\mathbf{M}_G AA$ with respect to the parameter p are obtained. After calculating the derivatives of the sub-matrices, discussed above, they can be substituted into equations (10) and (12) to obtain the derivatives of the dynamically condensed stiffness and mass matrices, respectively.

2.3. *Fast single parameter optimal modification* (*FASPO*) *method*

This method is applicable to those types of problems where a small shift in natural frequency is required and a cut and try method would be a time consuming process. Further, bigger shifts in a natural frequency by this method may give rise to unacceptable errors. The procedure is as follows. The first step is to make a sensitivity table for all the modification parameters of interest (lumped mass, depth/width of beam element in this case). This is determined by computation of derivatives of an objective (natural frequency in this case) with respect to modification parameters described above. These derivatives

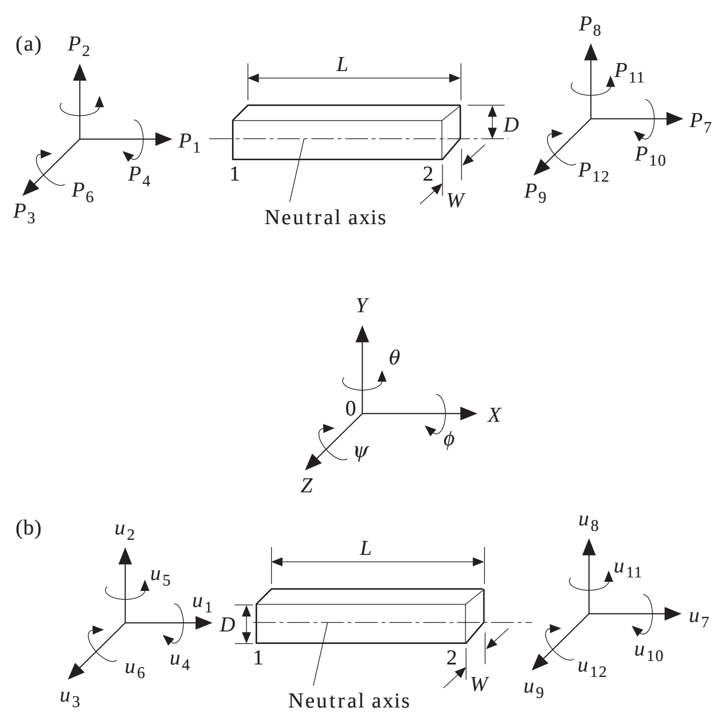

Figure 1. Co-ordinate system for a beam element: (a) positive force sign convention, (b) positive displacement sign convention.

are made dimensionless by obtaining c_{rj}, called the Pertinent Sensitivity Coefficient (PSC), from the equation

$$c_{rj} = (\partial \Omega_r / \partial p_j) \frac{p_j}{\Omega_r}, \tag{16}$$

where Ω_r is natural frequency of rth mode that is desired to be shifted and p is the modification parameter viz. mass or stiffness. The pertinent sensitivity coefficient can be utilized to make a database for all the parameters which are considered for modification of the system. The derivative $\partial \Omega_r / \partial p_j$ for a vibrating system whose spatial parameters, namely [**M**] and [**K**], are known have been established [6] and can be obtained from the equation

$$\partial \Omega_r / \partial p_j = 1/2\Omega_r \{\phi_r\}^T [\partial [\mathbf{K}] / \partial p_j - \Omega_r^2 \partial [\mathbf{M}] / \partial p_j] \{\phi_r\}. \tag{17}$$

Based on the PSC's of equation (16) the most effective parameter is selected which leads to the optimum shift of the desired natural frequency. This modifying parameter is quantified with the help of a frequency versus parameter graph. A typical frequency versus parameter graph is shown in Figure 2. In the figure, Ω_0 is the original natural frequency; p_0 the original modification parameter; Ω_d the desired natural frequency; Ω' the modified natural frequency and Δp is the predicted change in the modification parameter. For multi-degree-of-freedom systems, in general, this graph is non-linear in nature for the modification parameter p (in this case depth of beam element). It is observed from Figure 2 that for the desired frequency Ω_d, the change in the parameter p can be predicted as

$$\Delta p = (\Omega_d - \Omega_0)/(\partial \Omega / \partial p), \tag{18}$$

where $\partial \Omega / \partial p$ is the slope of the tangent drawn at the point A (Figure 2) corresponding to the unmodified natural frequency. Mathematically, the slope $\partial \Omega / \partial p$ represents the sensitivity derivative. In case several pertinent sensitivity coefficients (PSCs) are numerically equal, an order of preference table is prepared on the basis of a choice criterion. One such criterion may be based on the visual appeal of new modifications. There can be several such criteria which cannot be quantified but depend upon the practical constraints imposed by the actual design problem. In those cases where such constraints are non-existent, any of the parameters having an equal PSC value may be selected.

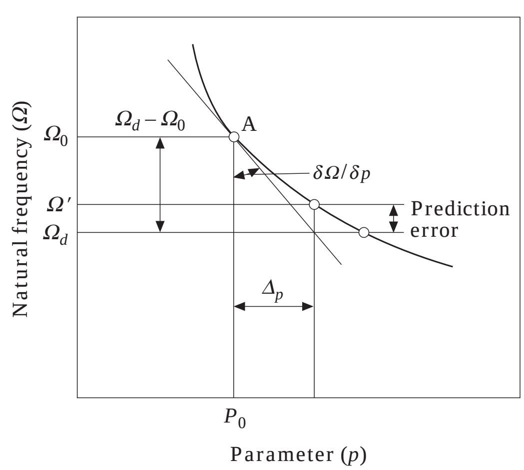

Figure 2. Fast single parameter optimization method.

2.4. *High accuracy single parameter optimization* (*HASPO*) *method*

Using the method of the last section the error would exceed the acceptable limits when shift in natural frequency is high. For these type of problems a generalisation of the original method is proposed which gives any desired value of natural frequency and quantifies the parameter precisely with high accuracy provided the design modification is feasible. This technique is again a single parameter based optimization method. The selection of the parameter p for modification proceeds on similar lines as described in the previous section. Other steps of this method are as follows.

Equation (18) can be rewritten as

$$\Delta p = p_1 - p_0 = \Omega_d - \Omega_0/\partial\Omega/\partial p \qquad \text{or} \qquad p_1 = p_0 + \frac{\Omega_d - \Omega_0}{\partial\Omega/\partial p}, \tag{19}$$

where p_1 is modified value of parameter p and p_0 is the original value of the parameter p.

Using equation (19) p_1 is obtained. This p_1 is used to get a new value of the natural frequency which is named Ω_1. The values of p_1 and Ω_1 are substituted back in equation (19) in place of p_0 and Ω_0 respectively. This gives again a new value of p_1; likewise the process is continued until the value of natural frequency becomes equal to the desired natural frequency i.e., Ω_d. In general, equation (19) can be written for kth iteration as

$$p^k = p^{k-1} + (\Omega_d - \Omega^{k-1})/(\partial\Omega/\partial p). \tag{20}$$

The above described process of convergence is also graphically shown in Figure 3. The computation is terminated after the iterative improvement in successive steps becomes smaller than the value of the desired accuracy in the desired objective which in this case is the natural frequency. The termination condition can be mathematically expressed as

$$(\Omega_d - \Omega^k)/\Omega_d > \text{desired accuracy in shifted } \Omega. \tag{21}$$

It is observed that any desired tolerance can be obtained by this method. Further, any amount of shift of natural frequency is possible, the only limitation is that the computed value of p should be feasible and acceptable.

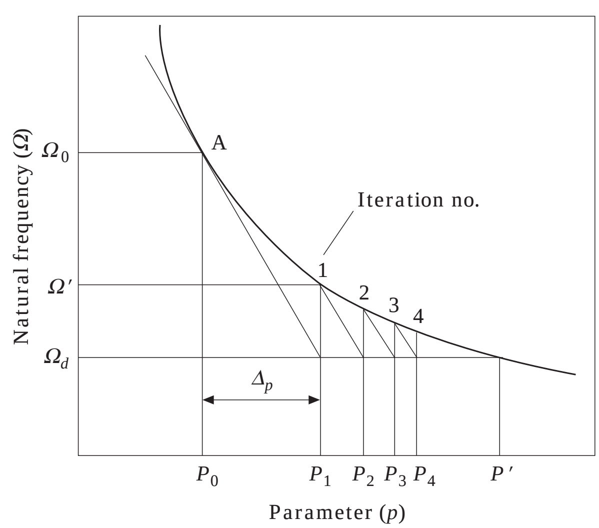

Figure 3. High accuracy parameter optimization method.

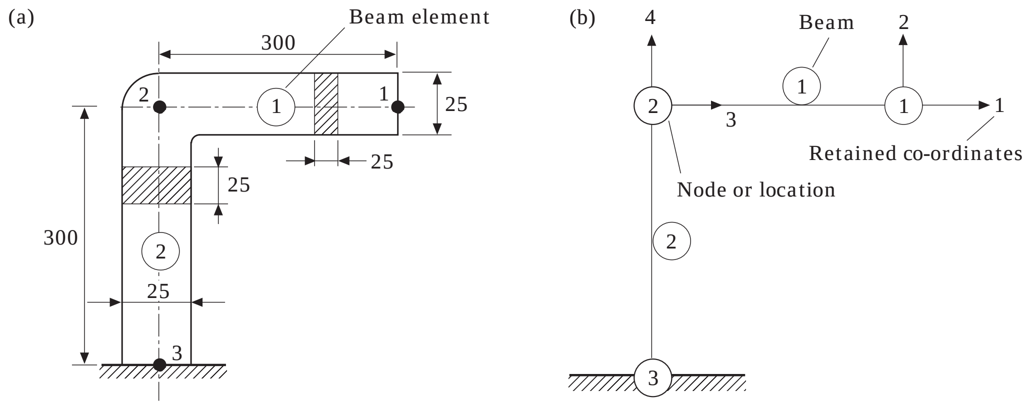

Figure 4. Simulated representation of the Γ-structure: (a) actual structure, (b) simulated representation. Dimensions are in mm.

3. NUMERICAL EXAMPLE

The proposed method for obtaining derivatives of dynamically condensed matrices $[\mathbf{K}_G]^R$ and $[\mathbf{M}_G]^R$ of a structure is illustrated with an example of a Γ-structure shown in Figure 4. The derivatives of beam element stiffness and mass matrices with respect to several design parameters can be stored as database. Using these elemental matrices, $\partial[\mathbf{K}_G]^R/\partial p$ and $\partial[\mathbf{M}_G]^R/\partial p$ are constructed for the Γ-structure by using equations (14) and (15) respectively. The derivatives of sub-matrices are also constructed. The design parameters p are taken as depth (D) and width (W) of the beams of which the Γ-structure is modelled. The derivatives of the system matrices can be obtained by substituting the sub-matrices above constructed into equations (10) and (12).

The above example problem is used to illustrate the optimal algorithms proposed in subsections 2.3 and 2.4. The FASPO and HASPO methods were applied to a simple finite element model (FEM) of a Γ-structure consisting of two beam elements shown in Figure 4. Young's modulus for the material of the beams of the Γ-structure is assumed to be $0{\cdot}207 \times 10^{12}$ N/m^2 and mass density is taken to be 7806 kg/m^3. A three-dimensional FEM

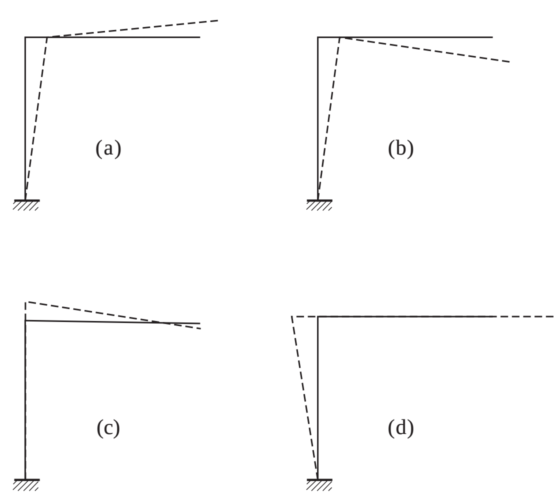

Figure 5. Mode shapes and natural frequencies of the Γ-structure: (a) first mode, $\Omega_1 = 451{\cdot}7$ rad/s; (b) second mode, $\Omega_2 = 1191{\cdot}2$ rad/s; (c) third mode, $\Omega_3 = 21\,123{\cdot}6$ rad/s; (d) fourth mode, $\Omega_4 = 44\,165{\cdot}4$ rad/s.

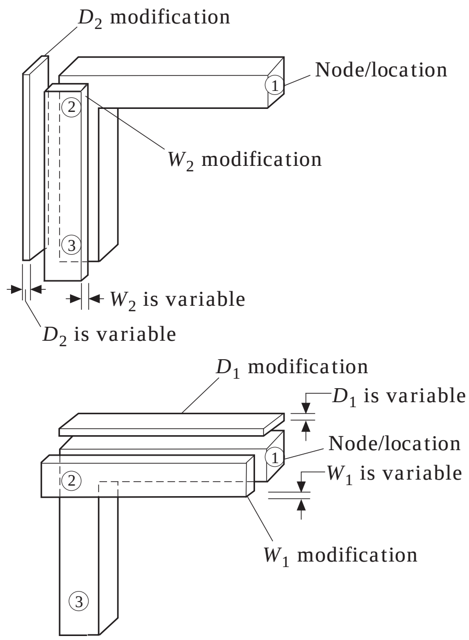

Figure 6. Beam modifications in the Γ-structure.

of the Γ-structure was taken to have a total of eighteen degrees of freedom. Six co-ordinates are grounded at location 3. Out of the remaining twelve co-ordinates, the four co-ordinates which are assumed to be of interest and are called "retained" co-ordinates as shown (Figure 4). The stiffness matrix and the consistent mass matrix neglecting shear deformations were developed. Dynamic condensation was used to reduce the size of matrices to the four "retained" co-ordinates of interest. Natural frequencies and the corresponding mass normalized eigenvectors for typical dimensions of the Γ-structure (see Figures 4) were analytically obtained and are given below.

The four natural frequencies are 451·7, 1191·2, 21 123·6, 44 165·4 rad/s and the corresponding mass normalized eigenvectors (mode shapes) are shown in Figure 5. The modification parameters D_j and W_j for the beam additions are shown in Figure 6. Modification parameters m_1 and m_2 correspond to lumped mass additions at node location 1 and 2 respectively shown in the figure. The PSCs and sensitivity derivatives for various modification parameters obtained by using equations (16) and (17) respectively are shown in Table 1. The parameter selected for modification is D_2 because it has the highest

TABLE 1

System parameters of Γ-structure and their sensitivities for first natural frequency

Parameter p	Sensitivity derivatives $\partial\Omega_r/\partial p$	PSC
D_1	−5773·0	−0·3188
D_2	23 822·9	1·3190
W_1	−7171·8	−0·3969
W_2	7171·6	0·3969

TABLE 2

Successive iterations for shifting first natural frequency of the Γ-structure from 451·7 rad/s to 800 rad/s

Iter. No. (k)	Modified Freq. (Ω_k)	$\Delta p_k = p_k - p_0$ here of ΔD_2 in m
0	451·70	0·0
1	778·00	0·014617
2	795·48	0·01554
3	799·00	0·01573
4	799·78	0·015772
5	799·95	0·015781
6	799·99	0·015783

sensitivity. The sensitivity derivatives of reduced mass and stiffness matrices are calculated using equations (10) and (12) respectively and substituted in equation (17) to find sensitivity derivatives of natural frequencies and tabulated as described above.

3.1. *FASPO method*

By letting it be desired that first natural frequency is to be shifted from 451·7 rad/s to 600 rad/s, the change in D_2 parameter may be predicted using equation (18) as $\Delta D_2 = (600 - 451{\cdot}7)/23\,822{\cdot}9 = 148{\cdot}3/23\,822{\cdot}9$. $\Delta D_2 = 0{\cdot}0062$ m, i.e., $D_2' = 0{\cdot}025 + 0{\cdot}0062 = 0{\cdot}0312$ m. The actual value will be 598·5 rad/s. Therefore, the error (1.5 rad/s) obtained is within the acceptable limit.

3.2 *HASPO method*

By letting it be desired that the first natural frequency is shifted to 800 rad/s, the change in D_2 will be predicted as 0·014617 m and the so obtained value would be 778 rad/s. This may not be acceptable. Therefore, the second method is used.

In this example a typically large value of shifting first natural frequency of the Γ-structure from 451·7 rad/s to 800 rad/s is assumed. The iterations are shown in Table 2. A typical accuracy is chosen which was achieved in just six iterations. Therefore, the desired change in natural frequency can be obtained by adding a beam element of 0·015783 m at D_2 as shown in Figure 6.

4. CONCLUDING REMARKS

Two sensitivity based optimal beam element modification methods have been developed. The first method implements a small shift in natural frequency and is achieved by modifying the most sensitive single design parameter viz. one dimension of beam element. The predicted modification shifts the natural frequency within acceptable accuracies. This method, based on the calculation of the derivative of natural frequency, estimates efficiently the effectiveness of the dynamic behaviour of a system to changes in system parameters and modifies the desired natural frequency in one modification cycle. The second method also employs single parameter modifications but the iterative procedure employed in this method can predict the desired modification within any degree of accuracy even for large shifts of natural frequency. The optimal modification algorithms have been implemented on a simple vibrating system.

5. FUTURE RESEARCH

The obvious need is the development of updated/corrected analytical models of structures on which structural dynamic modification and optimization has to be implemented. The analytical models such as finite element models need to be updated/corrected in the light of incomplete measured data. The corrected model is expected to predict accurately the dynamic behaviour of the structure. The techniques to develop updated analytical models are necessary to provide a suitable database for structural dynamic modification and optimization discussed in this study. While efforts are under way in the direction of updating finite element models, this work develops the techniques necessary to undertake beam element type modifications in physical spaces by employing an accurate database of system matrices (stiffness and mass) of the structure.

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